

ROBOT'S HAND AND EXPANSIONS IN NON-INTEGER BASES

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ABSTRACT. We study a robot hand model in the framework of the theory of expansions in non-integer bases. We investigate the reachable set, we introduce a grasping model and we study some grasping configurations.

1. INTRODUCTION

Aim of this paper is to give a model of a robot's hand based on the theory of expansions in non-integer bases. Self-similarity of configurations and an arbitrarily large number of fingers (including the opposable thumb) and phalanges are the main features. Binary controls rule the dynamics of the hand, in particular the extension and the rotation of each phalanx. We investigate the reachable set of this hand and we introduce a grasping model, i.e., we investigate the capability of the robot hand of grasping objects and keeping them stable. We refer to [BK00] for a detailed survey on the modelization of the grasping problem and related features, like equilibrium, stability and optimality of a grasping configuration. In [LP81] the kinematics of a manipulating hand is described in a detailed way. In the last ten years, also in view of the recent technical developments, the researchers explored the self-configuring properties of robotic hands in order to get an automatic motion-planning. Thanks to sensors, like cameras [KMA01, AHE01], and tactile sensors [KKU02], the context-awareness capabilities of robots were exploited: modern robots are able to explore the geometry of the target object and to self-configure in order to grasp it.

All the models we mentioned so far share the assumption that the physical properties of the robot hand devices are fixed: there is a fixed number of fingers and each finger has a fixed number of phalanges. In our model, thanks to the extension control, the number of phalanges is arbitrary: this gives a possibly large and dense reachable set and, at the same time, easily adapts to no-collision tasks because “hulky” phalanges can be retracted. The self-similar structure of every finger gives access to iterated function systems and fractal geometry theories in order to study reachability and control algorithms. Robotic devices studied in a similar fashion can be found in [MED96].

Some of our theoretical tools come from the theory of non-integer bases. For an overview on this topic we refer to [Rén57], [Par60], [EK98] and to the book [DK02]. In particular, expansions in non-integer bases were introduced

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in [R  n57]. For the geometrical aspects of the expansions in complex base namely the arguments that are more related to our problem, we refer to [Knu60], [Pen65],[GG79], [Gil81],[Gil87],[ABB⁺04],[AT04] and to [IKR92].

We now describe in a more detailed way the setting of our paper. Our robot hand is composed by an arbitrary number of fingers, including the opposable thumb. Each finger moves on a plane. Every plane is assumed to be parallel to the others, excepting the thumb and the index finger, that belong to the same plane.

A discrete dynamical system models the position of the extremal junction of every finger. A *configuration* is a sequence of states of the system corresponding to a particular choice for the controls, while the union of all the possible states of the system is named *reachable set for the finger*. The closure of the reachable set is named *asymptotic reachable set*. Our model includes two binary control parameters on every phalanx of every finger of the robot hand. The first control parameter rules the length of the phalanx, that can be either 0 or a fixed value, while the other control rules the angle between the current phalanx and the previous one. Such an angle can be either π , namely the phalanx is consecutive to the previous, or a fixed angle $\pi - \omega \in (0, \pi)$.

The dynamics of the finger ensures the set of possible configurations to be self-similar. In particular the sub-configurations can be looked at as scaled miniatures with constant ratio ρ , named *scaling factor*, of the whole structure. We use this property to investigate a class graspable objects: under appropriate assumptions, if an object is graspable, then every properly scaled, rotated and translated copy of this object is also graspable.

The paper is organized as follows. In Section 2 we introduce the model and in Section 3 we remark its relation with the theory of non-integer number systems. Self-similarity of the configurations and some reachability results are also shown. In Section 4 we discuss a necessary and sufficient condition to avoid self-intersecting configurations in a particular case. The grasping problem and some grasping configurations are finally discussed in Section 5.

2. THE MODEL

In our model the robot hand is composed by H fingers, every finger has an arbitrary number of phalanxes. We assume junctions and phalanxes of each finger to be thin, so to be respectively approximated with their middle axes and barycentres and we also assume the junctions of every finger to be coplanar. Inspired by the human hand, we set the fingers of our robot as follows: the first two fingers are coplanar and they have in common their first junction (they are our robotic version of the thumb and the index finger of the human hand) while the remaining $H - 1$ fingers belong to parallel planes whose distance is a fixed δ .

We now describe in more detail the model of a robot finger. A configuration of a finger is the sequence $(x_k)_{k=0}^K \subset \mathbb{R}^3$ of its junctions. The configurations of every finger are ruled by two phalanx-at-phalanx motions: extension and rotation. In particular, the length of k -th phalanx of the finger is either 0 or $\frac{1}{\rho^k}$, where $\rho > 1$ is a fixed ratio: this choice is ruled by the

a binary control we denote by using the symbol u_k , so that the length l_k of the k -th phalanx is

$$l_k := \|x_k - x_{k-1}\| = \frac{u_k}{\rho^k}.$$

As all the phalanxes of a finger belong to the same plane, say p , in order to describe the angle between two consecutive phalanxes, say the $k-1$ -th and the k -th phalanx, we just need to consider a one-dimensional parameter, ω_k . Each phalanx can lay on the same line as the former or it can form with it a fixed planar angle $\omega \in (0, \pi)$, whose vertex is the $k-1$ -th junction. In other words, two consecutive phalanxes form either the angle π or $\pi - \omega$. By introducing the binary control v_k we have that the angle between the $k-1$ -th and k -th phalanx is $\pi - \omega_k$, where

$$\omega_k = v_k \omega.$$

To give a clear description the dynamics of a finger, say the h -th finger of the hand, we start by simple configurations and we gradually generalize them. Assume the h -th finger to be oriented along a direction $\mathbf{v}^{(h)}$ parallel to the plane $p^{(h)}$. Consider now $A_\omega^{(h)}$, the clockwise rotation matrix with angle ω about the axis $r^{(h)}$ that is the unitary vector normal to $p^{(h)}$.

Example 1. If $p^{(h)} : z = 0$ then $r^{(h)} = (0, 0, 1)^T$ and

$$A_\omega^{(h)} = \begin{pmatrix} \cos \omega & \sin \omega & 0 \\ -\sin \omega & \cos \omega & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Remark 2. In general, $A_0^{(h)}$ is the three-dimensional unit matrix. By well-known properties of the rotation matrices, we also have

$$(1) \quad A_{\omega_1}^{(h)} \cdot A_{\omega_2}^{(h)} = A_{\omega_1 + \omega_2}^{(h)}.$$

In the *full-rotation* and *full-extension* configuration, namely when both the rotation control and the extension control are constantly equal to one, we have

$$(2) \quad \begin{cases} \mathbf{x}_1^{(h)} = \mathbf{x}_0^{(h)} + \frac{1}{\rho} A_\omega^{(h)} \mathbf{v}^{(h)} \\ \mathbf{x}_{k+1}^{(h)} = \mathbf{x}_k^{(h)} + \frac{1}{\rho} A_\omega^{(h)} (\mathbf{x}_k^{(h)} - \mathbf{x}_{k-1}^{(h)}) \end{cases}.$$

By introducing the rotation control parameter v_k in the above equation, we get the general full-extension dynamics

$$(3) \quad \begin{cases} \mathbf{x}_1^{(h)} = \mathbf{x}_0^{(h)} + \frac{1}{\rho} (v_1 A_\omega^{(h)} + (1 - v_1) A_0^{(h)}) \mathbf{v}^{(h)} \\ \mathbf{x}_{k+1}^{(h)} = \mathbf{x}_k^{(h)} + \frac{1}{\rho} (v_k A_\omega^{(h)} + (1 - v_k) A_0^{(h)}) (\mathbf{x}_k^{(h)} - \mathbf{x}_{k-1}^{(h)}) \end{cases}$$

Recalling the notation $\omega_k = \omega v_k$ we may rewrite (3) in the more compact form

$$(4) \quad \begin{cases} \mathbf{x}_1^{(h)} = \mathbf{x}_0^{(h)} + \frac{1}{\rho} A_{\omega_k}^{(h)} \mathbf{v}^{(h)} \\ \mathbf{x}_{k+1}^{(h)} = \mathbf{x}_k^{(h)} + \frac{1}{\rho} A_{\omega_k}^{(h)} (\mathbf{x}_k^{(h)} - \mathbf{x}_{k-1}^{(h)}) \end{cases}.$$

Example 3. *Following Example 1, set*

$$\begin{aligned} p^{(h)} &: z = 0; \\ \mathbf{x}_0^{(h)} &:= (0, 0, 0)^T; \\ \mathbf{v}^{(h)} &:= (1, 0, 0)^T; \\ \rho &> 1; \\ u_1 = v_1 = u_2 = v_2 &= 1; \end{aligned}$$

we have

$$\mathbf{x}_1^{(h)} = \mathbf{x}_0^{(h)} + \frac{1}{\rho} A_\omega^{(h)} \mathbf{d}^{(h)} = \left(\frac{\cos \omega}{\rho}, -\frac{\sin \omega}{\rho}, 0 \right)^T;$$

$$\begin{aligned} \mathbf{x}_2^{(h)} &= \mathbf{x}_1^{(h)} + \frac{1}{\rho} A_\omega^{(h)} (\mathbf{x}_1^{(h)} - \mathbf{x}_0^{(h)}) \\ &= \mathbf{x}_1^{(h)} + \frac{1}{\rho^2} A_{2\omega}^{(h)} \mathbf{d}^{(h)} \\ &= \left(\frac{\cos \omega}{\rho} + \frac{\cos 2\omega}{\rho^2}, -\frac{\sin \omega}{\rho} - \frac{\sin 2\omega}{\rho^2}, 0 \right)^T. \end{aligned}$$

Also remark

$$\mathbf{x}_2^{(h)} = (\Re(z), \Im(z), 0)^T$$

where

$$z = \sum_{j=1}^2 \frac{1}{(\rho e^{i\omega})^j}.$$

We now deal the non-full-rotation case, i.e., we describe what happens when some phalanx does not extend because the related extension control is zero. In a non-rotation configuration, namely every rotation control v_k is 0, all the phalanxes are parallel to $\mathbf{v}^{(h)}$ and we have

$$(5) \quad \begin{cases} \mathbf{x}_1^{(h)} = \mathbf{x}_0^{(h)} + \frac{u_1}{\rho} \mathbf{v}^{(h)} \\ \mathbf{x}_{k+1}^{(h)} = \mathbf{x}_k^{(h)} + \frac{\rho u_{k+1}}{\rho} (\mathbf{x}_k^{(h)} - \mathbf{x}_{k-1}^{(h)}) \end{cases}$$

and the closed formula

$$(6) \quad \mathbf{x}_k^{(h)} = \mathbf{x}_0^{(h)} + \sum_{j=1}^k \frac{u_j}{\rho^j} \mathbf{v}^{(h)}.$$

Example 4. *Set*

$$\begin{aligned} p^{(h)} : z &= \frac{x}{2} + \frac{y}{2} + 1; \\ \mathbf{x}_0^{(h)} &:= (0, 0, 1)^T \in p^{(h)}; \\ \mathbf{v}^{(h)} &:= \left(0, \frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}}\right)^T \parallel p^{(h)}; \\ \rho &= \sqrt{5}; \\ u_1 &= u_2 = 1; \end{aligned}$$

we have

$$\begin{aligned} \mathbf{x}_1^{(h)} &= \mathbf{x}_0^{(h)} + \frac{u_1}{\rho} \mathbf{v}^{(h)} = \left(0, \frac{2}{5}, \frac{6}{5}\right) \in p^{(h)} \\ \mathbf{x}_2^{(h)} &= \mathbf{x}_1^{(h)} + \frac{u_2}{\rho} (\mathbf{x}_1^{(h)} - \mathbf{x}_0^{(h)}) \mathbf{v}^{(h)} \\ &= \left(0, \frac{2}{5} \left(1 + \frac{1}{\sqrt{5}}\right), 1 + \frac{1}{5} \left(1 + \frac{1}{\sqrt{5}}\right)\right) \in p^{(h)} \end{aligned}$$

and also

$$\mathbf{x}_2^{(h)} = \mathbf{x}_0^{(h)} + \left(\frac{1}{\rho} + \frac{1}{\rho^2}\right) \mathbf{v}^{(h)}.$$

It is left to discuss the case $u_{k+1} = 0$ and $v_{k+1} = 1$, namely when the $k + 1$ -phalanx does not extend but a rotation is transmitted. The position of the extremal junction of the k -th phalanx $\mathbf{x}_k^{(h)}$ and $\mathbf{x}_{k+1}^{(h)}$ coincide because no extension is performed, but something in the internal state of the system changes: an “invisible” rotation is determined by the control $v_{k+1} = 1$.

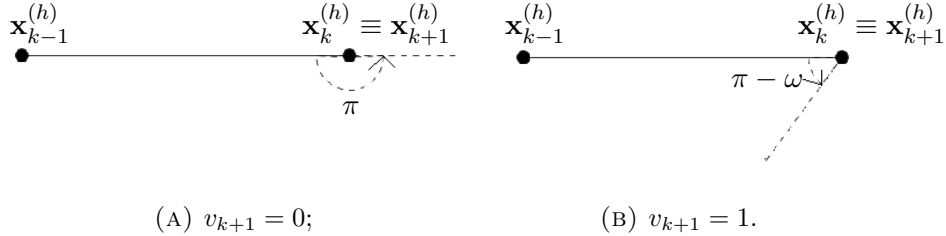


FIGURE 1. In both cases $u_{k+1} = 0$.

In particular the model keeps memory of the choice of v_{k+1} affecting the subsequent rotations (see Figures 1 and 2.). An invisible rotation becomes effective when a subsequent phalanx is extended. The number of no-extension control is not a priori fixed and, consequently, a fixed memory

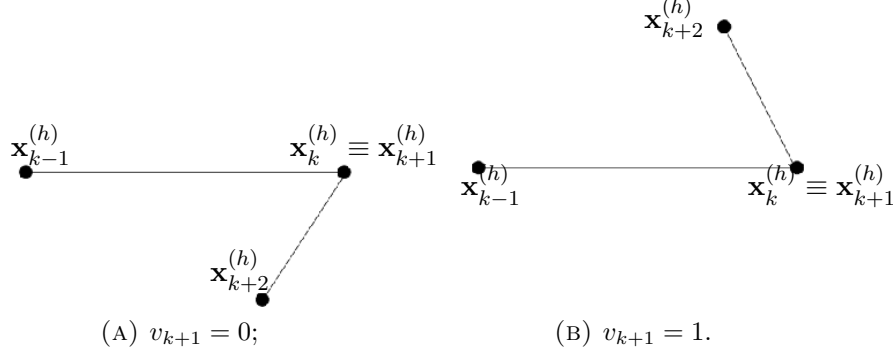


FIGURE 2. In both cases $u_{k+1} = 0$, $u_{k+2} = 1$ and $v_{k+2} = 1$.

storage for this information is not possible. In terms of dynamical equations, this means that the a purely recursive relation of the form

$$\mathbf{x}_{k+1}^{(h)} = f(\mathbf{x}_k^{(h)}, \mathbf{x}_{k-1}^{(h)}, \dots, \mathbf{x}_{k-j}^{(h)}, u_{k+1}, v_{k+1})$$

for some fixed $j \in \{0, \dots, k\}$ and some $f : \mathbb{R}^{j+1} \times \{0, 1\} \times \{0, 1\} \rightarrow \mathbb{R}^3$, can not fully describe the state of the system: the history of the “invisible” rotations needs to be stored somewhere else.

To introduce the general equation of our model, i.e., a relation between junctions also encompassing the “rotate but not extend” case, we recursively define $\{E_k\}$, the sequence of orthonormal matrices representing a local system of coordinates whose origin is $\mathbf{x}_k^{(h)}$.

Remark 5. The coordinates $\mathbf{x}' \in \mathbb{R}^3$ of a point x in the system with matrix base E_k and origin x_k are

$$\mathbf{x}' = E_k^{-1} \mathbf{x} - \mathbf{x}_k^{(h)}.$$

The system of coordinates E_k rotates of an angle ω if the rotation control v_k equals to 1, i.e.,

$$(7) \quad E_k = A_{\omega_k}^{(h)} E_{k-1}$$

E_0 represents the orientation of the initial phalanx, in particular its origin is x_0 , its Oxy plane is $p^{(h)}$ and its x -axes oriented along the direction $\mathbf{v}^{(h)}$. Setting $\mathbf{e}_1 := (1, 0, 0)^T$ we also have

$$(8) \quad E_0 \mathbf{e}_1 = \mathbf{v}^{(h)}.$$

By construction, the coordinates of $\mathbf{x}_{k+1}^{(h)}$ in the system whose basis matrix is E_k and whose origin is in $\mathbf{x}_k^{(h)}$ are

$$A_{\omega_{k+1}}^{(h)} E_k \cdot \left(\frac{u_{k+1}}{\rho^{k+1}}, 0, 0 \right)^T = E_{k+1} \cdot \left(\frac{u_{k+1}}{\rho^{k+1}}, 0, 0 \right)^T.$$

The dynamics of the extremal junction of the finger is hence obtained by changing the local coordinates of $\mathbf{x}_{k+1}^{(h)}$ in to the global ones:

$$(9) \quad \mathbf{x}_{k+1}^{(h)} = \mathbf{x}_k^{(h)} + \frac{u_{k+1}}{\rho^{k+1}} E_{k+1} \mathbf{e}_1.$$

In view of Remark 2, we have

$$E_{k+1} = A_{\omega_{k+1}}^{(h)} E_k = A_{\omega_{k+1}}^{(h)} A_{\omega_k}^{(h)} \cdots A_{\omega_1}^{(h)} E_0 = A_{\Omega_k}^{(h)} E_0,$$

where

$$(10) \quad \Omega_k := \sum_{n=1}^k \omega_n = \sum_{n=1}^k v_n \omega.$$

Thus, also in view of (8) we get the general equation

$$(11) \quad \mathbf{x}_{k+1}^{(h)} = \sum_{j=1}^{k+1} \frac{u_j}{\rho^j} A_{\Omega_j}^{(h)} \mathbf{v}^{(h)}.$$

2.1. Reachability. A point $\mathbf{x} \in \mathbb{R}^3$ is called *reachable by the h -th finger* if there exists a couple of control sequences $(u_j)_{j=1}^k$ and $(v_j)_{j=1}^k$ such that

$$(12) \quad \mathbf{x} = \sum_{j=1}^{k+1} \frac{u_j}{\rho^j} A_{\Omega_j}^{(h)} \mathbf{v}^{(h)}.$$

The point x is said to be *asymptotically reachable by the h -th finger* if it satisfies

$$(13) \quad \mathbf{x} = \sum_{j=1}^{\infty} \frac{u_j}{\rho^j} A_{\Omega_j}^{(h)} \mathbf{v}^{(h)}$$

for some couple of infinite control sequences $((u_j)_{j \geq 1}, (v_j)_{j \geq 1})$. We use the symbols $R^{(h)}$ and $R_{\infty}^{(h)}$ to respectively denote set of reachable and asymptotically reachable points with respect the h -th finger of the hand. In particular

$$R^{(h)} = \left\{ \sum_{j=1}^k \frac{u_j}{\rho^j} A_{\Omega_j}^{(h)} \mathbf{v}^{(h)} \mid \Omega_j = \sum_{n=1}^j v_n \omega; \ u_j, v_n \in \{0, 1\}; \ k \in \mathbb{N} \right\};$$

$$R_{\infty}^{(h)} = \left\{ \sum_{j=1}^{\infty} \frac{u_j}{\rho^j} A_{\Omega_j}^{(h)} \mathbf{v}^{(h)} \mid \Omega_j = \sum_{n=1}^j v_n \omega; \ u_j, v_n \in \{0, 1\} \right\}.$$

We also define $R_k^{(h)}$ the set of reachable points in time k , so that

$$R_k^{(h)} = \left\{ \sum_{j=1}^k \frac{u_j}{\rho^j} A_{\Omega_j}^{(h)} \mathbf{v}^{(h)} \mid \Omega_j = \sum_{n=1}^j v_n \omega; \ u_j, v_n \in \{0, 1\} \right\};;$$

Remark 6. *The following relations hold*

$$(14) \quad R^{(h)} = \bigcup_{k=0}^{\infty} R_k^{(h)};$$

$$(15) \quad R_{\infty}^{(h)} = \overline{R^{(h)}};$$

in particular for every asymptotically reachable point there exists an arbitrarily close reachable point.

Finally we call *reachable* (resp. *asymptotically reachable*) a point $\mathbf{x} \in \mathbb{R}^3$ that is reached (resp. *asymptotically reached*) by any finger of the hand, i.e., if there exists $h = 0, \dots, H$ such that

$$\begin{aligned} \mathbf{x} &\in R^{(h)}; \\ (\text{resp. } \mathbf{x} &\in R_\infty^{(h)};) \end{aligned}$$

The set of reachable points is

$$R := \bigcup_{h=0}^H R^{(h)}$$

the set of asymptotically reachable points is

$$R_\infty := \bigcup_{h=0}^H R_\infty^{(h)}$$

Remark 7. As we assumed all the phalanxes of a fixed finger to be coplanar, we have

$$R \subset \bigcup_{h=1}^H p^{(h)},$$

where $p^{(h)}$ is the plane of the h -finger. We remark that there are only H distinct planes because we assumed the first two fingers, the thumb and the forefinger, to belong to the same plane $p^{(1)}$.

3. ROBOT'S HAND AND EXPANSIONS IN COMPLEX BASES

In this section we discuss reachability and grasping conditions in the framework of expansions in complex bases. Given a complex number λ greater than 1 in modulus and a possibly infinite set $A \subset \mathbb{C}$ we say that $z \in \mathbb{C}$ is *representable* in *base* λ and with *alphabet* A if there exists a sequence $(z_j)_{j \geq 1}$ of digits of A such that

$$z = \sum_{j=1}^{\infty} \frac{z_j}{\lambda^j}.$$

A digit sequence $(z_j)_{j \geq 1}$ satisfying the above equality is called *expansion* of z in base λ and with alphabet A . It is well known coplanar rotations, like the ones performed by each finger of our hand, can be read as products on the complex plane and, consequently, to perform infinite rotations and scalings (like in the case of asymptotic reachability problem) equals to consider complex-based power series and, consequently, expansions in non-integer bases. In what follows we formalize this concept. By choosing an appropriate global coordinate system, we may assume without loss of generality that

- the hand has $H + 1$ fingers, the first one and the second one laying on the plane $p^{(1)} : z = 0$;
- the remaining $H - 1$ fingers lay on parallel planes $p^{(h)} : z = \delta(h - 1)$, for $h = 2, \dots, H$ and some $\delta > 0$;

- each finger has an arbitrary number of phalanxes, whose ratio is $\rho^{(h)} > 1$ for $h = 0, \dots, H$;
- the angle between two consecutive phalanxes of the h -th finger is either 0 or $\pi - \omega^{(h)}$:

With these settings, the normal vector $r^{(h)}$ to the plane $p^{(h)}$ is constantly $(0, 0, 1)^T$ and the rotation matrix $A_{\omega^{(h)}}^{(h)}$ is

$$(16) \quad A_{\omega^{(h)}}^{(h)} = \begin{pmatrix} \cos(\omega^{(h)}) & \sin(\omega^{(h)}) & 0 \\ -\sin(\omega^{(h)}) & \cos(\omega^{(h)}) & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Remark 8. By employing the isometry between \mathbb{R}^2 and \mathbb{C} , we have for every $\mathbf{d} = (d_1, d_2, d_3)^T \in \mathbb{R}^3$ the relation

$$A_{\omega^{(h)}}^{(h)} \cdot \mathbf{d} = (\Re(z), \Im(z), d_3)^T$$

where

$$z = e^{-i\omega^{(h)}}(d_1 + id_2) \in \mathbb{C}.$$

We now consider $A_{\Omega_j}^{(h)}$ where $\Omega_j = \sum_{n=1}^j v_n \omega^{(h)}$, for some (binary) rotation control sequence $(v_n)_{n=1}^j$. Similarly to the case discussed in Remark 8, we have for every $\mathbf{d} = (d_1, d_2, d_3)^T \in \mathbb{R}^3$

$$A_{\Omega_j}^{(h)} \cdot \mathbf{d} = (\Re(z), \Im(z), d_3)^T$$

where

$$z = e^{-i\Omega_j}(d_1 + id_2) \in \mathbb{C}.$$

Consequently, the reachability set $R^{(h)}$, containing all the reachable configurations of the h -finger, satisfies

$$R^{(h)} = \left\{ (\Re(z), \Im(z), \delta(h-1)) \mid z = \sum_{j=1}^k \frac{u_j}{\rho^j} e^{-i\Omega_j}; u_j, v_n \in \{0, 1\}; k \in \mathbb{N} \right\}$$

and the study of the reachability, asymptotic or not, becomes equivalent to the study of the sets of complex numbers

$$\mathcal{R}^{(h)} := \left\{ \sum_{j=1}^k \frac{u_j}{\rho^j} e^{-i\Omega_j}; u_j, v_n \in \{0, 1\}; k \in \mathbb{N} \right\},$$

$$\mathcal{R}_k^{(h)} := \left\{ \sum_{j=1}^k \frac{u_j}{\rho^j} e^{-i\Omega_j}; u_j, v_n \in \{0, 1\} \right\}$$

and

$$\mathcal{R}_\infty^{(h)} := \left\{ \sum_{j=1}^\infty \frac{u_j}{\rho^j} e^{-i\Omega_j}; u_j, v_n \in \{0, 1\} \right\}.$$

Now, set $\lambda := \rho e^{i\omega^{(h)}}$ and consider the digit set $A_j := \{0, e^{i(j\omega - \Omega_j)}\} = \{0, e^{iN\omega}, N \in \{0, 1, \dots, j\}\}$, we have

$$\mathcal{R}_\infty^{(h)} = \left\{ \sum_{j=1}^\infty \frac{z_j}{\lambda^j} \mid z_j \in A_j; j \in \mathbb{N} \right\}.$$

and, setting, $A := \bigcup_j A_j$

$$\mathcal{R}_\infty^{(h)} \subseteq \left\{ \sum_{j=1}^{\infty} \frac{z_j}{\lambda^j} \mid z_j \in A; j \in \mathbb{N} \right\}.$$

consequently the asymptotic reachable set of a finger is a subset of the points whose two first coordinates are the real and imaginary part of representable numbers in base λ and with alphabet A .

Remark 9. If $\omega^{(h)} = \frac{p}{q}2\pi$ for some $p, q \in \mathbb{Z}$ then A is a finite set.

If we restrict to full-rotation configurations, namely when the rotation controls are constantly equal to 1, we have

$$\Omega_j = \sum_{n=1}^j v_n \omega^{(h)} = j\omega$$

and, consequently, $A = A_j = \{0, 1\}$ for every j .

In the case of full-extension configurations A does not contain 0.

4. SOME FEATURES OF THE ASYMPTOTIC REACHABLE SET

4.1. Self-similarity. It is well known that all the sets of representable numbers with positional numbers system are self-similar, in particular they are the unique fixed point of appropriate linear iterated function systems. We recall that an iterated function system (IFS) is a finite set of contractive functions $f_j : \mathbb{C} \rightarrow \mathbb{C}$. Every IFS admits a fixed point, which is unique, with respect to the *Hutchinson operator*

$$\mathcal{F} : S \mapsto \bigcup_{j=1}^J f_j(S)$$

In particular there exists a set $R \subseteq \mathbb{C}$ s.t. for every $S \subseteq \mathbb{C}$

$$\lim_{k \rightarrow \infty} \mathcal{F}^k(S) = R,$$

i.e. R is the attractor of \mathcal{F} .

Example 10. The middle third Cantor set is the set of representable numbers in base 3 and with alphabet $\{0, 2\}$ and it is the attractor of the IFS $\{f_1(x) = \frac{x}{3}; f_2(x) = \frac{x+2}{3}\}$.

The asymptotic reachable set has this property, as well. In particular we have the following result, whose proof can be found in [LL11].

Proposition 11. For every $\rho > 1$ and $\omega \in (0, \pi)$, the approximated reachability set $\mathcal{R}_\infty^{(h)}(\rho, \omega)$ is the (unique) fixed point of the IFS

$$(17) \quad \mathcal{F}_{\rho, \omega} = \{f_h : \mathbb{C} \rightarrow \mathbb{C} \mid h = 1, \dots, 4\}$$

where

$$(18) \quad \begin{aligned} f_1 : x &\mapsto \frac{1}{\rho}x & f_2 : x &\mapsto \frac{e^{-i(\pi-\omega)}}{\rho}x \\ f_3 : x &\mapsto \frac{1}{\rho}(x+1) & f_4 : x &\mapsto \frac{e^{-i(\pi-\omega)}}{\rho}(x+1). \end{aligned}$$

4.2. Reachability. As we noticed in Remark 9, the set of points that can be asymptotically reached with full-rotation configurations corresponds to the set of representable numbers with a suitable base and with alphabet $A = \{0, 1\}$, in particular

$$\mathcal{R}^{(h),fr} := \left\{ \sum_{j=1}^{\infty} \frac{z_j}{\lambda^j} \mid z_j \in \{0, 1\} \right\} \subset \mathcal{R}^{(h)}.$$

This gives access to several results on complex-based representability that can be adapted to our case. Set $\mathbb{C}_R := \{z \in \mathbb{C} \mid |z| \leq R\}$ with $R > 0$ and $\omega \in (0, 2\pi)$. In [KL07] is shown that if ρ is sufficiently close to 1 then every complex number $z \in \mathbb{C}_R$ has at least one expansion in base $\lambda = \rho e^{i\omega}$ and with alphabet $\{0, 1\}$, i.e., every point in \mathbb{C}_R can be reached by a full-rotation configuration. Moreover if $\omega = \frac{1}{q}2\pi$, with $q \in \mathbb{N}$ and $q \geq 3$, and $\rho \leq 2^{1/q}$, then $\mathcal{R}^{(h),fr}$ is a polygon with $2q$ edges if q is odd and q edges otherwise [Lai].

We also remark that in [AT00] and [AT05] can be found a study of the topological properties of the so-called *fundamental domains* of two-dimensional expansions: our set $\mathcal{R}^{(h),fr}$ is a particular case of such domains. We refer to [AT04] as a survey on this argument.

5. HOW TO AVOID SELF-INTERSECTING CONFIGURATIONS: A PARTICULAR CASE

In general, the dynamics of the fingers does not prevent self-intersecting configurations (see Figure 3) and, clearly, this needs to be avoided in order to keep the physical sense of the model. In this section we show sufficient conditions to avoid self-intersecting configurations in a particular case, namely when the angle between phalanxes is $\pi/3$, i.e., $\omega = 2\pi/3$. Our starting point is next result, whose proof can be found in [LL11].

Lemma 12. *For every $\rho > 1$*

$$(19) \quad \text{conv}(\mathcal{R}_{\infty}^{(h)}(\rho, 2\pi/3)) = \text{conv}(\{\mathbf{v}_1(\rho), \mathbf{v}_2(\rho), \mathbf{v}_3(\rho), \mathbf{v}_4(\rho)\})$$

where

$$\begin{aligned} \mathbf{v}_1(\rho) &= \frac{1}{\rho - 1}, & \mathbf{v}_2(\rho) &= \frac{e^{-i\frac{2\pi}{3}}}{\rho - 1}, \\ \mathbf{v}_3(\rho) &= \frac{e^{-\frac{2\pi}{3}i}}{\rho} + \frac{e^{-i\frac{4\pi}{3}}}{\rho(\rho - 1)}, & \mathbf{v}_4(\rho) &= \frac{e^{-i\frac{4\pi}{3}}}{\rho(\rho - 1)}. \end{aligned}$$

Theorem 13. *If the angle between the phalanxes is $\pi/3$ and if the ratio is $\rho \geq 2$, then all the configurations are not self-intersecting.*

Proof. A configuration is a finite subsequence of junctions of a finger on the complex plane and, consequently, every configuration is a scaled and rotated copy of a configuration starting from the initial junction x_0 and with the first extended phalanx parallel to the real axis (see Proposition 11). Therefore we may consider without loss of generality only configurations of the form $(x_j)_{0 \leq j < \infty}$ with $x_0 = 0$ and with $x_1 = \frac{1}{\rho}$ and it suffices to prove that any subsequent phalanx, namely the segment joining two consecutive junctions,

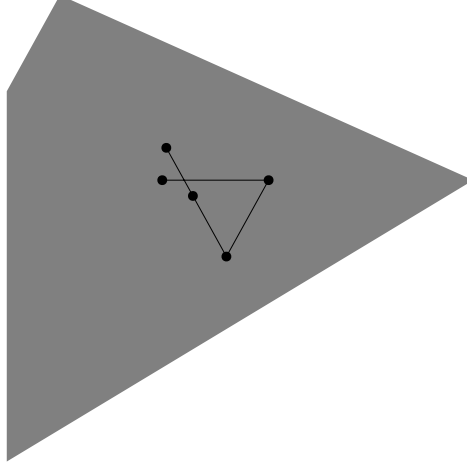


FIGURE 3. A self-intersecting configuration with $\rho = 1.5$ and $\omega = \pi/3$

does not intersect the first one. Let J be the smallest integer such that $x_J \neq x_1$. We have

$$x_J = x_1 + \frac{1}{\rho^J} e^{-i \sum_{n=1}^J v_n \pi/3} = x_1 + \frac{1}{\rho^J} e^{-iN\pi/3}$$

for some $N \in \{0, 1, 2\}$. The asymptotic reachable set from x_J is hence the following

$$\mathcal{R}_\infty^{(h)}(x_J) = x_J + \frac{1}{\rho^J} e^{-iN\pi/3} \mathcal{R}_\infty^{(h)}$$

and, in view of Lemma 12,

$$\text{conv}(\mathcal{R}_\infty^{(h)}(x_J)) = \text{conv}(\{x_J + \frac{1}{\rho^J} e^{-iN\pi/3} \mathbf{v}_j(\rho) \mid j = 1, \dots, 4\}).$$

By a direct computation, we have that the intersection between $\text{conv}(\mathcal{R}_\infty^{(h)}(x_J))$ and the first phalanx $\{x_0 + tx_1 \mid t \in [0, 1]\}$ is empty if $\rho > 2$. In particular, if $\rho > 2$ and if $N = 0$ then the real part of every reachable point is greater than $\frac{1}{\rho}$, namely greater than one of the endpoints of the phalanx, if $N = 1$ or $N = 2$ then the imaginary part of any reachable point is respectively strictly smaller or greater than 0. When $\rho = 2$ only infinite full-extension configurations intersect the first phalanx. Since configurations are finite sequences, this proves the “if part” of the theorem. If $\rho < 2$ then a direct computation shows that the configuration generated by the control sequences $(u_j)_{j=1}^J$ and $(v_j)_{j=1}^J$ with $u_j = 1$ for every $j = 1, \dots, J$ and $v_1 = 0$, $v_2 = v_3 = 1$ and $v_j = 0$ for every $j = 3, \dots, J$ is self-intersecting for every sufficiently large J (see Figure 3). \square

6. GRASPING PROBLEM

Grasping problem consists in finding surfaces on an object and a configuration of the robot's hand satisfying the following three conditions [LP81]

- the hand must be in contact with the object;
- the configuration must be reachable;
- the object must be stable once grasped, i.e., it must not slip during subsequent motions.

Our stability requirement is the equilibrium between the forces and the moments acting on the object. We assume the contact to be frictionless, i.e., the finger can only exert a force along the common normal at the point of contact. In our model, the modulus of this force can be set in order to grasp an object. In particular the behavior of each phalanx is ruled by the triplet (u_k, v_k, α_k) , where $u_k, v_k \in \{0, 1\}$ are the usual extension and rotation controls, while $\alpha_k \in [0, 1]$ rules the modulus of the force. We say that an object is *graspable* if there exists a finite control sequence such that the resulting configuration satisfies the grasping conditions (contact constraints, reachability, stability). Such a configuration is called *grasping configuration*.

6.1. Grasping a circle: primary circle and primary grasping configuration. We are interested in describing some graspable objects, by introducing an explicit grasping configuration. We focus on the ability of a finger to grasp a particular circle using its first phalanxes and we use self-similarity to extend this result to smaller and farther from the origin graspable circles.

Let $\omega \in (0, \pi)$ be the angle of rotation of a finger and let J_ω be the smallest integer such that

$$(20) \quad \omega(J_\omega - 1) < \pi \leq \omega J_\omega \quad \text{mod } 2\pi$$

Example 14. If $\omega = 2\pi/3$ then $J_\omega = 2$.

Consider a configuration whose motion controls satisfy for every $0 \leq j \leq J$, for some $J \geq J_\omega$,

$$u_j = \begin{cases} 1 & \text{if } j = 1, J_\omega, J_\omega + 1; \\ 0 & \text{otherwise;} \end{cases} \quad v_j = \begin{cases} 1 & \text{if } j \leq J_\omega + 1; \\ 0 & \text{otherwise.} \end{cases}$$

By construction, if $\omega J_\omega \neq \pi$ then the prolongations of the phalanxes in configuration form a triangle, see Figure 4.A. If the contact points between the inscribed circle and the triangle belong to the phalanxes and not to the prolongations, namely if the phalanxes are long enough, then we call it *primary grasped circle*. If otherwise $\omega J_\omega = \pi$, then every extended phalanx but the second one is parallel to the first phalanx, see Figure 4.B. In this case the primary grasped circle is the (unique) circle tangent to three distinct extended phalanxes: in particular, it is the circle inscribed in the rhombus whose edges are as long as the J_ω -th phalanx and whose internal angles are ω and $\pi - \omega$. The configuration admitting a primary grasped circle with the smallest number of extended phalanxes is called *primary grasping configuration*.

Next result establishes necessary and sufficient conditions for the existence of the primary grasped circle.

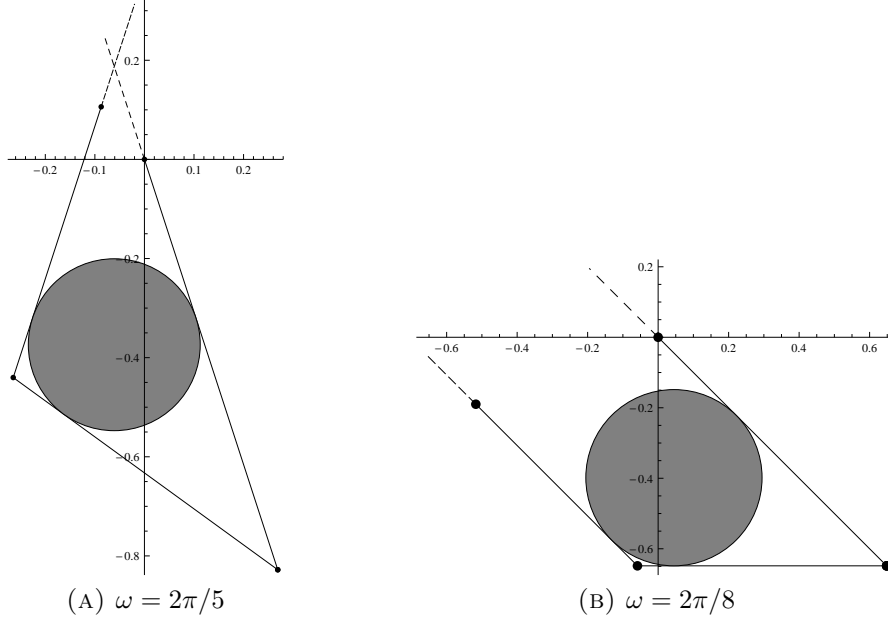


FIGURE 4. Primary grasping configurations and primary grasped circles.

Theorem 15. *For every $\omega \in (0, \pi)$, there exists a primary grasped circle if and only if*

$$(21) \quad \rho < 2 + \tan(\omega(J_\omega - 1)/2) \cot(\omega/2).$$

Proof. Consider a grasping configuration of length $J \geq J_\omega$. The extended phalanxes are the first one and every phalanx between the J_ω -th and J -th ones. Moreover all phalanxes between the $J_\omega+1$ -th and the last belong to the same line, because their rotation controls are constantly 0. Hence we may construct a circle tangent to the prolongations of these phalanxes and we call the tangent points C_1 , C_{J_ω} and $C_{J_\omega+1}$. We need to establish necessary and sufficient conditions on ρ to have C_1 , C_{J_ω} and $C_{J_\omega+1}$ respectively belonging to the first, to the J_ω -th and to any subsequent phalanx. Remark that the J_ω -th phalanx shares both its endpoints with other phalanxes, hence the prolongations we are considering only refer to the first and to the last phalanxes: by construction C_{J_ω} is always tangent to the J_ω -th phalanx. In particular we have that the distance between C_{J_ω} and the J_ω -th junction is lower than the length of the J_ω -th phalanx:

$$(22) \quad |x_1 - C_{J_\omega}| \leq \frac{1}{\rho^{J_\omega}}.$$

Now, C_1 belongs to the first phalanx if and only if

$$(23) \quad |x_1 - C_1| \leq \frac{1}{\rho}$$

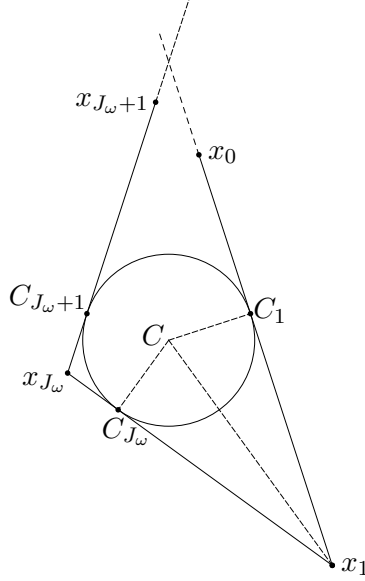


FIGURE 5. The triangle with vertices C , C_1 and x_1 and the triangle with vertices C , C_{J_ω} and x_1 are equal, because they are right triangles with a common edge and with two equal edges (the ray of the inscribed circle). Hence $|x_{J_\omega} - C_{J_\omega}| = |x_{J_\omega} - C_{J_\omega+1}|$. This also implies that the edge with endpoints x_1 and C is the bisector of the angle in x_1 .

where x_1 is the position of the first junction and $\frac{1}{\rho}$ is the length of the first phalanx. Similarly $C_{J_\omega+1}$ belongs to a phalanx if and only if

$$(24) \quad |x_{J_\omega} - C_{J_\omega+1}| < \frac{1}{\rho^{J_\omega}(\rho - 1)}$$

indeed the right-hand side of the above inequality is the upper bound of the length of a finite sequence of adjacent phalanxes. A classical result in plane geometry states that if we consider two consecutive edges of a polygon admitting an inscribed circle, then the distances between the related tangent points and the common vertex are equal (see Figure 5).

In our case this implies, together with (22),

$$(25) \quad |x_1 - C_1| = |x_1 - C_{J_\omega}| < \frac{1}{\rho}.$$

Similarly we may rewrite (24) as follows

$$(26) \quad |x_{J_\omega} - C_{J_\omega}| < \frac{1}{\rho^{J_\omega}(\rho - 1)}.$$

Now, the angle between the first two active phalanxes is $\pi - (J_\omega - 1)\omega$; therefore if we call r the ray of the inscribed circle we have

$$(27) \quad |x_{J_\omega} - C_{J_\omega}| = \frac{1}{\rho^{J_\omega}} - |x_1 - C_{J_\omega}| = \frac{1}{\rho^{J_\omega}} - r \left| \tan\left(\frac{\pi - (J_\omega - 1)\omega}{2}\right) \right|$$

(see Figure 5). Since the angle in the junction x_{J_ω} is $\pi - \omega$, we also have

$$(28) \quad |x_{J_\omega} - C_{J_\omega}| = r |\tan((\pi - \omega)/2)|.$$

By a comparison between (27) and (28) we deduce

$$|x_{J_\omega} - C_{J_\omega}| = \frac{1}{\rho^{J_\omega}} \frac{\tan(\omega/2)}{\tan(\omega/2) + \tan((J_\omega - 1)\omega/2)}.$$

and, finally, the equivalence between (24) and (21). \square

Example 16. *If $\omega = 2\pi/3$ then the primary grasped circle is well defined if $\rho < 3$.*

If $\omega = 2\pi/5$ then the primary grasped circle is well defined if $\rho < 2G + 3$, where $G = (1 + \sqrt{5})/2$ is the Golden Ratio.

Proposition 17. *The primary grasp configuration is a grasp for the primary grasping circle.*

Proof. By construction the primary grasp configuration satisfies both the contact and reachability constraints, hence we need to study the stability of the system. In a primary grasp configuration, there exist three contact points between the finger and the circle, C_1 , C_{J_ω} and $C_{J_\omega+1}$. The contact involves the first, the J_ω -th phalanx and a particular subsequent phalanx. We call \mathbf{n}_1 , \mathbf{n}_{J_ω} and $\mathbf{n}_{J_\omega+1}$ the external normal versors to the boundary of the circle in the contact points. Since the contact phalanxes are tangent to the circle, \mathbf{n}_1 , \mathbf{n}_{J_ω} and $\mathbf{n}_{J_\omega+1}$ are also normal to the first, the J_ω -th and to the $J_\omega + 1$ -th phalanx. This allows us to explicitly determine \mathbf{n}_1 , \mathbf{n}_{J_ω} and $\mathbf{n}_{J_\omega+1}$, in particular we may assume without loss of generality the finger to belong to the xy -plane in \mathbb{R}^3 and get

$$(29) \quad \begin{aligned} \mathbf{n}_1 &= (\Re(z_1), \Im(z_1), 0) \quad \text{where } z_1 = -e^{i(-\omega+\pi/2)}; \\ \mathbf{n}_{J_\omega} &= (\Re(z_{J_\omega}), \Im(z_{J_\omega}), 0) \quad \text{where } z_{J_\omega} = -e^{i(-J_\omega\omega+\pi/2)}; \\ \mathbf{n}_{J_\omega+1} &= (\Re(z_{J_\omega+1}), \Im(z_{J_\omega+1}), 0) \quad \text{where } z_{J_\omega+1} = e^{i(-(J_\omega+1)\omega+\pi/2)}. \end{aligned}$$

We call C the center of the circle and we assume it to coincide with its barycenter. Since the contact is frictionless, the forces acting on the grasped circle are hence the following

$$(30) \quad \begin{aligned} F_1 &= \alpha_1 \mathbf{n}_1; \\ F_{J_\omega} &= \alpha_{J_\omega} \mathbf{n}_{J_\omega}; \\ F_{J_\omega+1} &= \alpha_{J_\omega+1} \mathbf{n}_{J_\omega+1}. \end{aligned}$$

with $\alpha_1, \alpha_{J_\omega}, \alpha_{J_\omega+1} \in [0, 1]$.

We recall that our stability requirement is the static equilibrium, hence we need to show that we may properly control the moduli of F_1, F_{J_ω} and $F_{J_\omega+1}$ to have equilibrium between these forces

$$(31) \quad F_1 + F_{J_\omega} + F_{J_\omega+1} = 0$$

and between their moments

$$(32) \quad (C_1 - C) \times F_1 + (C_{J_\omega} - C) \times F_{J_\omega} + (C_{J_\omega+1} - C) \times F_{J_\omega+1} = 0.$$

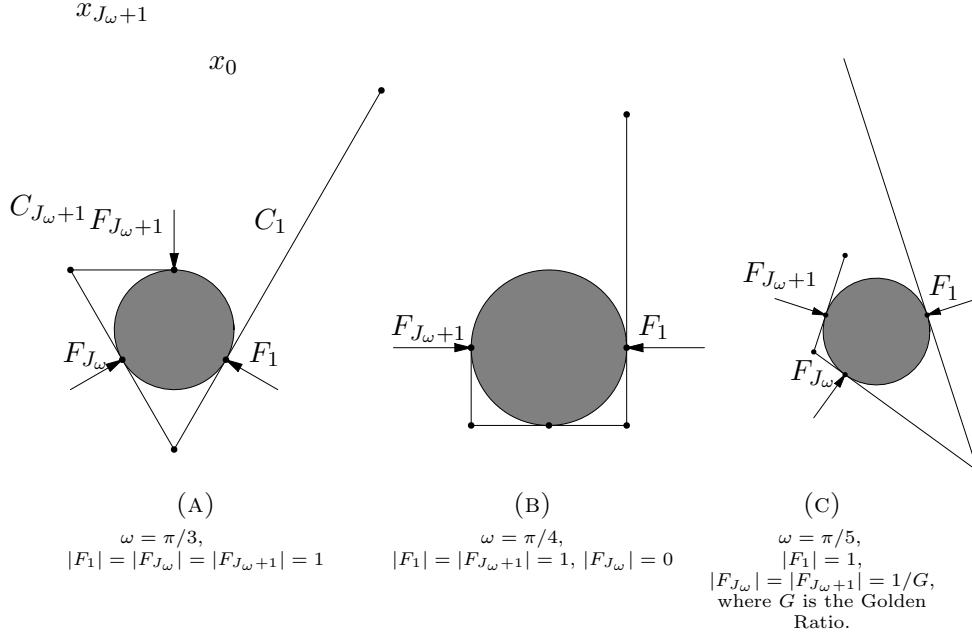


FIGURE 6

Now, (31) can be set in the complex plane, and in particular we obtain the complex equation

$$\alpha_1 e^{i(-\omega+\pi/2)} + \alpha_{J\omega} e^{i(-J\omega\omega+\pi/2)} + \alpha_{J\omega+1} e^{i(-(J\omega+1)\omega+\pi/2)} = 0$$

whose parametric solutions are

$$(33) \quad \begin{cases} \alpha_1(t) = t; \\ \alpha_{J\omega}(t) = -t \frac{\sin(\omega J\omega)}{\sin \omega}; \\ \alpha_{J\omega+1}(t) = t \frac{\sin((J\omega - 1)\omega)}{\sin \omega}; \end{cases}$$

with $t \in \mathbb{R}$. Setting

$$t = \begin{cases} 1 & \text{if } J\omega\omega = \pi; \\ \min \left\{ 1, -\frac{\sin \omega}{\sin(J\omega\omega)}, \frac{\sin \omega}{\sin((J\omega - 1)\omega)} \right\} & \text{otherwise,} \end{cases}$$

we have by, the definition of $J\omega$ and by the assumption $\omega \in (0, \pi)$, the corresponding solutions $\alpha_1, \alpha_{J\omega}$ and $\alpha_{J\omega+1}$ to belong to $[0, 1]$ and that they also satisfy (32). \square

6.2. Grasping and self-similarity. In Section 4.1 we recalled that the complex reachable set $\mathcal{R}^{(h)}$ is the attractor the IFS $\mathcal{F}_{\rho, \omega}$ defined in (17).

This implies that every reachable point can be obtained by an appropriate concatenation of the linear maps:

$$(34) \quad \begin{aligned} f_1 : x &\mapsto \frac{1}{\rho}x & f_2 : x &\mapsto \frac{e^{-i(\pi-\omega)}}{\frac{\rho}{\rho}}x \\ f_3 : x &\mapsto \frac{1}{\rho}(x+1) & f_4 : x &\mapsto \frac{e^{-i(\pi-\omega)}}{\rho}(x+1). \end{aligned}$$

To better understand the relation between this IFS and the control sequences, we introduce the map $d : \{0, 1\} \times \{0, 1\} \rightarrow \{1, 2, 3, 4\}$ such that

$$\begin{aligned} d(0, 0) &= 1 & d(0, 1) &= 2 \\ d(1, 0) &= 3 & d(1, 1) &= 4 \end{aligned}$$

and, fixing the motion control sequences $(u_j)_{j=1}^J$ and $(v_j)_{j=1}^J$, we define the index sequence

$$d_j := d(u_j, v_j)$$

for every $j = 1, \dots, J$. Then the reachable point

$$x_J := \sum_{j=1}^J \frac{u_j}{\rho^j} e^{-i \sum_{n=1}^j v_n \omega}$$

also satisfies the relation

$$(35) \quad x_J = f_{d_J} \circ f_{d_{J-1}} \circ \dots \circ f_{d_1}(0).$$

Since f_1, \dots, f_4 are invertible maps, we also have

$$(36) \quad 0 = f_{d_1}^{-1} \circ \dots \circ f_{d_{J-1}}^{-1} \circ f_{d_J}^{-1}(x_J).$$

Notation 18. For every $h = 1, 2, 3, 4$ we define the map from \mathbb{R}^2 onto itself

$$(37) \quad \mathbf{f}_h : (x, y) \mapsto (\Re(f_h(x + iy)), \Im(f_h(x + iy))).$$

The above reasonings are the basis of the following result, that is a necessary condition for a planar object to be grasped by a finger belonging to the xy -plane of \mathbb{R}^3 .

Proposition 19. Let $O \subset \mathbb{R}^2$ be an object grasped by a finger. Suppose J to be the smallest integer such that the J -th phalanx is in contact with O and let $\mathbf{x}_J \in \mathbb{R}^2$ be the corresponding junction, reached by the motion control sequences $(u_j)_{j=1}^J$ and $(v_j)_{j=1}^J$. Then the scaled, translated and rotated copy of O

$$(38) \quad \mathbf{f}_{d_1}^{-1} \circ \dots \circ \mathbf{f}_{d_{J-1}}^{-1} \circ \mathbf{f}_{d_J}^{-1}(O).$$

is graspable. Conversely, let O be a graspable object whose grasping configuration is given by the control sequences $(\bar{u}_j)_{j=1}^N$, $(\bar{v}_j)_{j=1}^N$ and $(\bar{\alpha}_j)_{j=1}^N$ and consider the motion control sequences $(u_j)_{j=1}^J$ and $(v_j)_{j=1}^J$. Then the control sequences $u_1, \dots, u_J, \bar{u}_1, \dots, \bar{u}_N, v_1, \dots, v_J, \bar{v}_1, \dots, \bar{v}_N$ and $\alpha_1, \dots, \alpha_J, \bar{\alpha}_1, \dots, \bar{\alpha}_J$, where $\alpha_j = 0$, yield a configuration satisfying the contact and stability constraints for the object

$$(39) \quad \mathbf{f}_{d_J} \circ \mathbf{f}_{d_{J-1}} \circ \dots \circ \mathbf{f}_{d_1}(O).$$

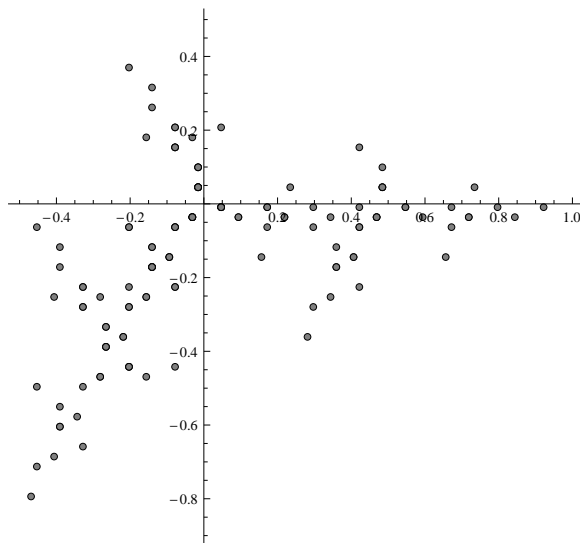


FIGURE 7. Every circle in this figure is of the form $\mathbf{f}_{d_1} \circ \dots \circ \mathbf{f}_{d_4}(O_{\rho,\omega})$ with $\omega = \pi/3$, $\rho = 2$ and $d_1, \dots, d_4 \in \{1, 2, 3, 4\}$. All the circles in (A) are graspable, indeed there exists a configuration satisfying reachability and stability requirements by Proposition 19 and in this case such a configuration also satisfies the contact constraints, namely it is grasping. Indeed the angle between two consecutive phalanxes is either π , $\pi/6$ or $5\pi/6$ then no phalanx can intersect the scaled copy of $O_{\rho,\omega}$ without intersecting another phalanx. But this case is excluded by Theorem 13.

Remark 20. *The configuration described in the second part of the above result is not in general grasping because it does not satisfy the contact constraints. Nevertheless in the case $2 \leq \rho < 3$ and $\omega = \pi/3$, namely when there not exist self-intersecting configurations (see Theorem 13) and the primary graspable circle is well defined (see Proposition 15 and Example 16), then a set of graspable circles can be constructed by iteratively applying the maps $\mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3$ and \mathbf{f}_4 on the primary grasped circle, see Figure 7-(A).*

6.3. Grasping three-dimensional objects: some examples. In this section we show two grasping configurations involving the whole hand. The general settings are the following: the hand has 5 fingers ($H = 5$), the angle between the phalanxes is constantly $\pi/2$, the scaling ratio is $\rho = 2$ for all the fingers. The distance between phalanxes is $\delta =$.

In the first example we are interested in grasping a cylinder whose axis is normal to the planes of the fingers and whose section O is a rescaling and a translation of the primary grasped circle, O_ω , in particular

$$(40) \quad O = \mathbf{f}_3(O_\omega)$$

where \mathbf{f}_3 is like in (37) – see also (34). Remark that O_ω is well defined by virtue of Proposition 15 and the related grasping configuration is $u_j = v_j = 1$ for $j = 1, 2, 3$ and $\alpha_1 = \alpha_3 = 1$ and $\alpha_2 = 0$. Since $d(1, 0) = 3$, by virtue of Proposition 7 then the configuration where the digit 1 is perpendic to

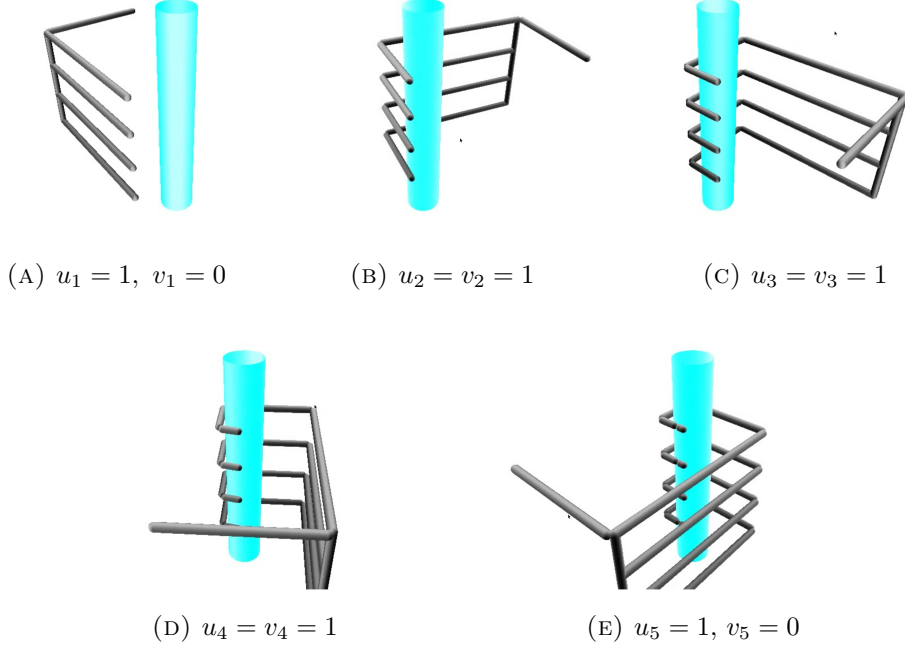


FIGURE 8. Various stages of the grasping of a cylinder, whose section is $\mathbf{f}_3(O_\omega)$. Due to numerical and graphical reasons, we extended one more phalanx with respect to the (minimal) primary grasping configuration, so that for every finger the resulting motion control sequence is $u_1 = u_4 = 1$, $v_1 = v_5 = 0$ and $u_j = v_j = 1$ with $j = 2, 3, 4$.

extension control sequence and the digit 0 is perpended to the rotation control sequence, is a suitable grasp for O – see (40). It is easy to verify that this configuration also satisfies the contact constraints and if we apply it to all the fingers of the hand we obtain a grasp for the cylinder whose section is equal to O – see Figure 8.

In Figure 9 we consider a cylinder whose section is the primary grasped circle after a rotation of $2\pi/3$ along the Oy direction. In this case we also use the opposable thumb, whose parameter ω is assumed to be different from the others, in particular $\omega^{(1)} = \pi/4$.

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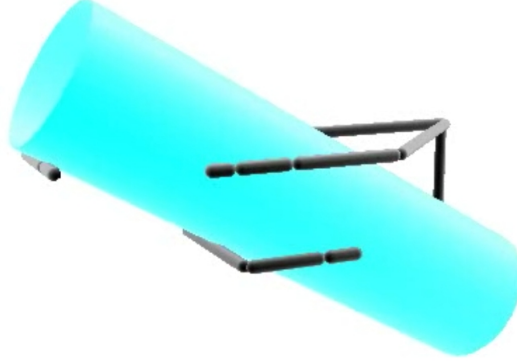


FIGURE 9. Thumb motion controls: $u_j = v_1 = v_2 = 1$ for $j = 1, 2, 3, 4$; $v_j = 0$ for $j = 3, 4$.
 Index finger motion controls: $u_j = v_3 = 1$ for $j = 1, 2, 3, 4$; $v_1, v_2, v_4 = 0$.
 Last finger motion controls $u_j = v_2 = v_3 = 1$ for $j = 1, 2, 3, 4$; $v_1, v_4 = 0$.

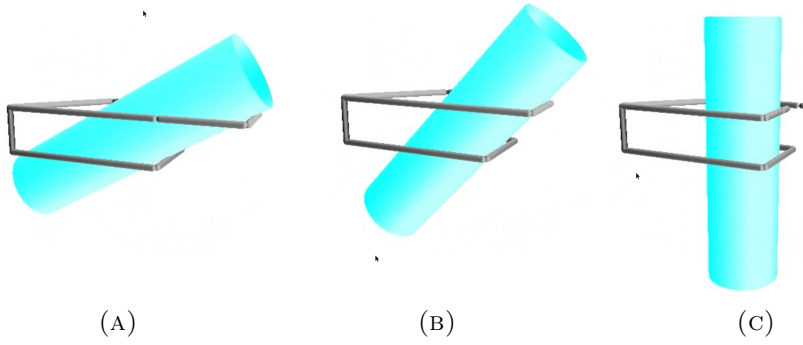


FIGURE 10. Numerical evidence suggests that an appropriate order in retracting phalanxes leads to a stable rotation of the cylinder

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